Dynamical Systems
Tutorial 9: Index Theory

June 10, 2019

1 Index Theory

Motivation: So far we learned to find the FPs, and their stability namely, we understand the dynamics in a small region around the FP. This is **local analysis**.

Index theory is a method that provides global information, that is, about a portion of the phase space which is not infinitesimal.

The type of questions index theory can answer are:
Must a closed trajectory always encircle a FP?
if so what type of FP are permitted?

Some **physical intuition**: Gauss's law - the electric flux through any closed surface is proportional to the electric charge within the closed surface:

$$\varepsilon_0 \Phi_E = q$$

or

$$\varepsilon_0 \oint_S E \cdot dA = q$$

The total charge in a region is computed by the integration of the flux of the electrical vector filed through surface.

So the **gauss surface** is replaced by general **curve**, the **electric field** is replaced by the **vector field** and the **charge** is replaced by the **index**.

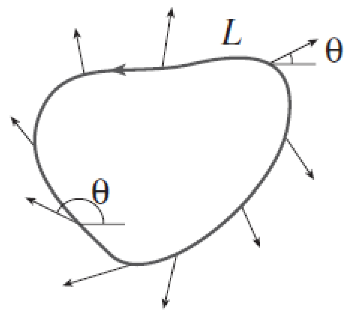


Figure 6.12. *Definition of the Poincaré index. Here $I_f(L) = 1$.*

$$\text{(6.1)} \quad \begin{aligned} \dot{x} &= P(x, y), \\ \dot{y} &= Q(x, y). \end{aligned}$$

We begin by defining the index of a *simple closed loop* L, a curve defined by a continuous, one-to-one mapping $L : \mathbb{S}^1 \to \mathbb{R}^2$ of the circle into the plane (recall §5.5). Such curves are often called *Jordan curves*. A simple example of such a mapping is $L = \{(\cos t, \sin t) : 0 \le t < 2\pi\}$, the unit circle. The loop L is assigned an orientation by the direction of its traversal; in this example the orientation is counterclockwise.

Taking $f = (P, Q)$ to be a vector field on $\mathbb{R}^2$ as in (6.1), define θ to be the direction of f so that $\tan\theta = Q/P$. The direction is well defined even when the slope is infinite, provided that P and Q do not simultaneously vanish—that is, everywhere except for the equilibria; see Figure 6.12. Using the direction field, Poincaré defined an index of L relative to the vector field.

> ▷ *Poincaré index*: Suppose $f \in C^0(\mathbb{R}^2, \mathbb{R}^2)$, L is an oriented, Jordan curve, and there are no equilibria of f on L. The index, $I_L(f)$, is the integer number of rotations of the vector $f(x)$ as x traverses the loop in the positive direction,
>
> $$I_L(f) \equiv \frac{\Delta\theta}{2\pi}, \tag{6.29}$$
>
> where $\Delta\theta$ is the net change in direction of f upon traversal of the loop.

When the vector field is C^1, $\Delta\theta$ can be obtained by integrating along the curve: $I_L(f) = \frac{1}{2\pi}\oint_L d\theta$. Given that $\tan\theta = Q/P$, we differentiate to obtain

$$\sec^2\theta d\theta = \frac{PdQ - QdP}{P^2}.$$

Since $\sec^2\theta = 1 + (Q/P)^2$, the index is then defined by the line integral

$$I_L(f) = \frac{1}{2\pi}\oint_L d\theta = \frac{1}{2\pi}\oint_L \frac{PdQ - QdP}{P^2 + Q^2}, \tag{6.30}$$

which can be evaluated explicitly if the loop L is given in parametric form.

Note that L is not a trajectory!!!

1.1 Properties

So assuming L has positive orientation, the index is the net number of counterclockwise revolutions around L made by the vector field. The following holds:

Lemma 1. *(6.5 in Meiss) If a curve L is deformed and does not cross an equilibrium, then $I_L(f)$ does not change. Similarly, if the curve is held fixed and the vector field is varied, then the index does not change as long as no equilibria fall on L throughout the deformation.*

So in particular homotopic curves have the same index as long as no equilibrium is crossed.

Lemma 2. *(6.6 in Meiss) If $f(x) = Ax$, where A is a nonsingular, 2×2 matrix and L is any counterclockwise loop enclosing the origin, then $I_L(f) = sgn(\det(A))$.*

Consequently, the index of a loop distinguishes between linear systems with saddle and non-saddle equilibria. We can also use it to detect the very existence of equilibria:

Theorem 1. *(6.7) If L is a Jordan curve and does not enclose an equilibrium of f, then $I_L(f) = 0$.*

Unfortunately, the reverse is not true - if $I_L(f) = 0$, we cannot conclude that there are no equilibria inside L. However, we may use the following result to determine if this is the case by refining the loop into subloops:

Lemma 3. *(6.8) The index of a sum of curves is the sum of the indices of the curves.*

1.2 Index of a fixed point

As you can probably already see, the index of a loop is closely tied to the fixed points this loop encloses. In a similar way, one can define the index of an equilibrium:

Definition 1. *Index of an isolated equilibrium: $I_{x^*}(f)$ is the index of any curve that encircles the equilibrium x^* and no others.*

According to Lemma 1 the index $I_{x^*}(f)$ is independent of the encircling loop, since the loop can be deformed to any other enclosing loop. Any loop that encloses a set of isolated equilibria can be partitioned into loops that enclose each individual equilibrium. Then Lemma 3 implies that the index of the original loop is the sum of the indices of the enclosed equilibria.

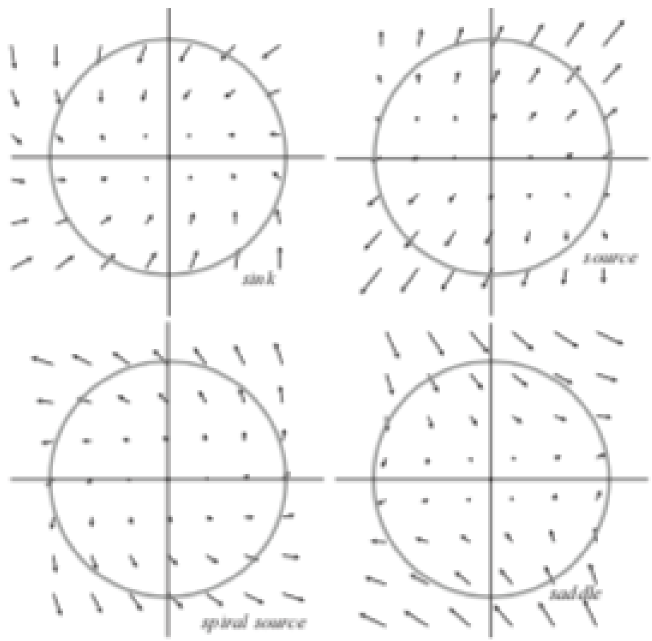


Figure 6.13. *Index of four types of hyperbolic matrices.*

It can be shown geometrically (see Meiss) that

1. The index of a sink, a source or a center is $+1$.
2. The index of a hyperbolic saddle point is -1.
3. The index of a closed orbit is $+1$.

In particular, the index can be used to detect the existence of periodic orbits:

Theorem 2. *(6.9 in Meiss) If γ is a periodic orbit of f, then $I_\gamma(f) = 1$.*

Corollary 1. *Any periodic orbit of f must enclose at least one equilibrium.*

If all the FPs within γ are hyperbolic, then there must in general be an odd number, $2n+1$, of which n are saddles and $n+1$ are either sinks or sources.

1.3 Examples

Example 1. *(Closed Curve - FP Free.)*

$$I(L) = \frac{1}{2\pi}\oint_L \frac{PdQ-QdP}{P^2+Q^2} = \left\{Green\text{'}s\ theorem\right\} =$$
$$\frac{1}{2\pi}\iint_U \left[\frac{\partial}{\partial P}\left(\frac{P}{P^2+Q^2}\right)+\frac{\partial}{\partial Q}\left(\frac{Q}{P^2+Q^2}\right)\right]dPdQ =$$
$$\frac{1}{2\pi}\iint_U \left[\left(\frac{(P^2+Q^2)-2P^2+(P^2+Q^2)-2Q^2}{(P^2+Q^2)^2}\right]dPdQ = 0$$

Example 2. *(Source)*
Given the source model:

$$\dot{x} = x$$
$$\dot{y} = y$$

Taking L as the circle loop around the FP at the origin.

$$I(0,0) = \frac{1}{2\pi}\oint_{x^2+y^2=1}\frac{xdy-ydx}{x^2+y^2} = \left\{ \begin{array}{ll} x=\cos(\theta) & dx=-\sin(\theta)d\theta \\ y=\sin(\theta) & dx=\cos(\theta)d\theta \end{array} \right\} =$$
$$\frac{1}{2\pi}\oint_{\theta\in S^1}(\cos^2(\theta)+\sin^2(\theta))d\theta = \frac{1}{2\pi}\oint_{\theta\in S^1} d\theta = 1$$

$$f(X) = A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\cdot\begin{pmatrix} x \\ y \end{pmatrix} \text{ so } \operatorname{sgn}\det A = 1$$

Example 3. *(Sink)*
Given the sink model:

$$\dot{x} = -x$$
$$\dot{y} = -y$$

Taking L as the circle loop around the FP at the origin.

$$I(0,0) = \frac{1}{2\pi}\oint_{x^2+y^2=1}\frac{(-x)(-dy)-(-y)(-dx)}{x^2+y^2} = \left\{ \begin{array}{ll} x=\cos(\theta) & dx=-\sin(\theta)d\theta \\ y=\sin(\theta) & dx=\cos(\theta)d\theta \end{array} \right\} =$$
$$\frac{1}{2\pi}\oint_{\theta\in S^1}(\cos^2(\theta)+\sin^2(\theta))d\theta = \frac{1}{2\pi}\oint_{\theta\in S^1} d\theta = 1$$

$$f(X) = A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\cdot\begin{pmatrix} x \\ y \end{pmatrix} \text{ so } \operatorname{sgn}\det A = 1$$

Example 4. *(Center)*
Given the center model:

$$\dot{x} = -y$$
$$\dot{y} = x$$

Taking L as the circle loop around the FP at the origin.

$$I(0,0) = \frac{1}{2\pi}\oint_{x^2+y^2=1} \frac{(-y)dx - x(-dy)}{x^2+y^2} = \left\{ \begin{array}{ll} x=\cos(\theta) & dx = -\sin(\theta)d\theta \\ y=\sin(\theta) & dx=\cos(\theta)d\theta \end{array} \right\} =$$
$$\frac{1}{2\pi}\oint_{\theta\in S^1}(\cos^2(\theta)+\sin^2(\theta))d\theta = \frac{1}{2\pi}\oint_{\theta\in S^1} d\theta = 1$$

$$f(X) = A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} \text{ so } \operatorname{sgn}\det A = 1$$

Example 5. *(Saddle)*
Given the saddle model:

$$\dot{x} = x$$
$$\dot{y} = -y$$

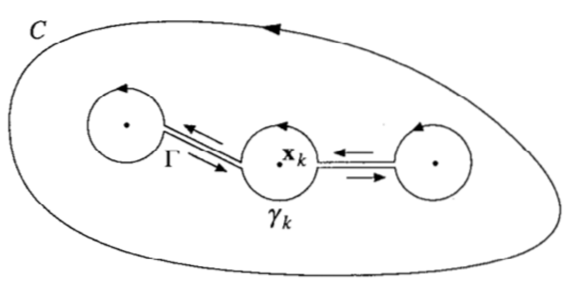
Taking L as the circle loop around the FP at the origin.

$$I(0,0) = \frac{1}{2\pi}\oint_{x^2+y^2=1} \frac{x(-dy)+ydx}{x^2+y^2} = \left\{ \begin{array}{ll} x=\cos(\theta) & dx = -\sin(\theta)d\theta \\ y=\sin(\theta) & dx=\cos(\theta)d\theta \end{array} \right\} =$$
$$-\frac{1}{2\pi}\oint_{\theta\in S^1}(\cos^2(\theta)+\sin^2(\theta))d\theta = \frac{1}{2\pi}\oint_{\theta\in S^1} d\theta = -1$$

$$f(X) = A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} \text{ so } \operatorname{sgn}\det A = -1$$

Example 6. *(proof of theorem 2)*
If we assume the flow direction on γ is positive (counterclockwise), sine f is tangent to γ it clearly makes a positive circuit as γ is traversed. Similarly, if the flow is in the opposite direction, then f still rotates once in a positive direction.

Example 7. *(closed curve has index as the sum of the indices of the enclosed FP)*
Use homotopy property of index to form almost cycle around FPs with narrow tunnel connecting of parallel lines which has same vectors so they cancel out, see Strogatz, page 179, figure 6.8.8.



Example 8. *(system)*

$$\dot{x} = xe^{-x}$$
$$\dot{y} = 1 + x + y^2$$

$\dot{x} = 0 \Rightarrow x = 0 \Rightarrow \dot{y} = 1 + y^2 > 0$. *So there is no FP in this system, by index theory* $\Rightarrow$ *there is no closed trajectory.*

Bibliography

- Meiss, J. D. (2007). Differential dynamical systems
- Strogatz, S. H. (1994) Nonlinear Dynamics and Chaos: With Applications to Physics, Biology, Chemistry, and Engineering